(12) FG

WVT-TR-76010

AD

ADA023902

THE STANDARD LINEAR MODEL IN A VISCOELASTIC LAMINATED COMPOSITE BEAM THEORY

APRIL 1976

BENET WEAPONS LABORATORY
WATERVLIET ARSENAL
WATERVLIET, N.Y. 12189

U.S. ARMY ARMAMENT COMMAND

TECHNICAL REPORT

AMCMS No. 611102.11.H4500.30

Pron No. EJ-6-Y0012-02-EJ-M7

APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED

DISCLAIMER

The findings in this report are not to be construed as an official Department of the Army position unless so designated by other authorized documents.

The use of trade name(s) and/or manufacturer(s) in this report does not constitute an official indorsement or approval.

DISPOSITION

Destroy this report when it is no longer needed. Do not return it to the originator.

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: WVT-TR-76010
2. GOVT ACCESSION NO.:
3. RECIPIENT'S CATALOG NUMBER:
4. TITLE (and Subtitle): The Standard Linear Model in a VISCOELASTIC [redacted] Laminated Composite Beam Theory
5. TYPE OF REPORT & PERIOD COVERED:
6. PERFORMING ORG. REPORT NUMBER:
7. AUTHOR(s): C. R. Thomas
8. CONTRACT OR GRANT NUMBER(s):
9. PERFORMING ORGANIZATION NAME AND ADDRESS:
Benet Weapons Laboratory
Watervliet Arsenal, Watervliet, N. Y. 12189
SARWV-RT
10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS:
AMCMS No. 611102.11.H4500.30
Pron No. EJ-6-Y0012-02-EJ-M7
11. CONTROLLING OFFICE NAME AND ADDRESS:
U.S. Army Armament Command
Rock Island, Illinois 61201
12. REPORT DATE: April 1976
13. NUMBER OF PAGES: 38
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office):
15. SECURITY CLASS. (of this report): UNCLASSIFIED
15a. DECLASSIFICATION/DOWNGRADING SCHEDULE:
16. DISTRIBUTION STATEMENT (of this Report):
Approved for public relase; distribution unlimited.
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report):
18. SUPPLEMENTARY NOTES:
19. KEY WORDS (Continue on reverse side if necessary and identify by block number):
Viscoelasticity
Laminates
Composite Materials
Beams (Structural)
Timoshenko Beam Theory
20. ABSTRACT (Continue on reverse side if necessary and identify by block number):

A microstructure or effective stiffness viscoelastic laminated beam theory is derived where individual reinforcing and matrix layers are modeled as viscoelastic Timoshenko beams. Kinetic, potential and dissipative energies are determined for individual beam layers modeled as viscoelastic Timoshenko beams with constitutive equations given by the standard linear model. The individual energies are summed over the layers in the beam, a smoothing operation is applied and an energy principle is applied to yield the equations of motion and boundary conditions for a viscoelastic laminated beam theory.

(14) WVT-TR-76010

AD

(6) THE STANDARD LINEAR MODEL IN A VISCOELASTIC LAMINATED COMPOSITE BEAM THEORY.

(10) C. R. THOMAS

(12) 47p.

(11) APR[redacted]76

BENET WEAPONS LABORATORY
WATERVLIET ARSENAL
WATERVLIET, N.Y. 12189

(9) TECHNICAL REP[redacted]T,

AMCMS No. 611102.11.H4500.30

(16) Pron [redacted] EJ-6-Y0012-02-EJ-M7

APPROVED FOR PUBLIC RELEASE. DISTRIBUTION UNLIMITED

371 050

TABLE OF CONTENTS

FIGURES

Reverse Page Blank

INTRODUCTION

A great deal of work has been accomplished in the area of elastic laminated effective stiffness or microstructure continuum theories and approximate plate and beam theories. By the same token, little has been accomplished with viscoelastic counterparts to these theories.

An elastic continuum theory which included effective stiffnesses for both the reinforcing and matrix layers of a laminated continuum was developed by Sun, Achenbach, and Herrmann [1, 2]. The continuum theory was utilized by Thomas [3] to study the simple thickness modes for laminated media with layering both parallel and perpendicular to the plate free surfaces. Sun [4] deduced a two dimensional theory for laminated plates from the three dimensional continuum theory. Velocity correction coefficients were introduced into the two dimensional theory by Thomas [5] and flexural and extensional vibrations for plate

[1] C. T. SUN, J. D. ACHENBACH and G. HERRMANN (1968) Journal of Applied Mechanics, 35, 467. Continuum Theory for a Laminated Medium.

[2] J. D. ACHENBACH, C. T. SUN and G. HERRMANN (1968) Journal of Applied Mechanics, 35, 689. On the Vibrations of a Laminated Body.

[3] C. R. THOMAS (1972) Journal of Sound and Vibration, 23(3), 341-361. Simple Thickness Modes for Laminated Composite Materials.

[4] C. T. SUN (1971) Journal of Applied Mechanics, 38, 231-238. Theory of Laminated Plates.

[5] C. R. THOMAS (1972) Journal of Sound and Vibration, 25(3), 407-431. Velocity Corrected Theory of Laminated Plates Applied to Free Plate Strip Vibrations.

strips and rectangular plates were studied by Thomas [6, 7] according to this theory and compared to similar results from effective modulus plate theories. A microstructure theory for an elastic, laminated composite beam was developed by Sun [8] and the approach utilized in this paper will be followed in deriving a viscoelastic, laminated composite beam theory. Thomas [9] showed that the flexure beam theory in reference [8] is directly obtainable through a simple reduction of the existing flexure equations for composite plates [4, 5].

A continuum theory for a viscoelastic laminated composite was developed by Grot and Achenbach [10], however the equations developed

---

[4] C. T. SUN (1971) Journal of Applied Mechanics, 38, 231-238. Theory of Laminated Plates.

[5] C. R. THOMAS (1972) Journal of Sound and Vibration, 25(3), 407-431. Velocity Corrected Theory of Laminated Plates Applied to Free Plate Strip Vibrations.

[6] C. R. THOMAS (1973) Journal of Sound and Vibration, 31(2), 195-211. Extensional Vibrations of Simply Supported Composite Plate Strips.

[7] C. R. THOMAS (1975) Journal of the Acoustical Society of America, 57(3), 655-659. Flexural and Extensional Vibrations of Simply Supported Laminated Rectangular Plates.

[8] C. T. SUN (1971) Journal of Applied Mechanics, 38, 947-954. Microstructure Theory for a Composite Beam.

[9] C. R. THOMAS (1973) Watervliet Arsenal Technical Report, R-WV-T-6-45-73. Flexure Equations of Motion for Laminated Composite Beams.

[10] R. A. GROT and J. D. ACHENBACH (1970) Acta Mechanica, 9, 245-263. Linear Isothermal Theory for a Viscoelastic Laminated Composite.

were not applied to any problems of wave propagation or vibration. It is certainly theoretically possible to start with the equations in reference [10], to make appropriate series expansions and derive a plate theory, and to then follow reference [9] to make a direct reduction to a viscoelastic beam theory. However, for convenience and simplicity of analysis, the approach in the current report will be to begin with the viscoelastic Timoshenko beam equations and work towards a viscoelastic laminated beam equation in the manner of reference [8]. With somewhat guarded conclusions, Stern, Bedford, and Yew [11] have demonstrated a definite need for an effective stiffness type formulation for viscoelastic laminates.

The current approach to obtaining a viscoelastic laminated beam theory will be a viscoelastic development which mirrors the elastic development given by Sun [8]. Surprisingly, the real difficulty is in obtaining the energies for a single layer modeled as a viscoelastic Timoshenko beam. The most pleasing and straightforward development of suitable viscoelastic Timoshenko beams results from a utilization

---

[8] C. T. SUN (1971) Journal of Applied Mechanics, 38, 947-954. Microstructure Theory for a Composite Beam.

[9] C. R. THOMAS (1973) Watervliet Arsenal Technical Report, R-WV-T-6-45-73. Flexure Equations of Motion for Laminated Composite Beams.

[10] R. A. GROT and J. D. ACHENBACH (1970) Acta Mechanica, 9, 245-263. Linear Isothermal Theory for a Viscoelastic Laminated Composite.

[11] M. STERN, A. BEDFORD, and C. H. YEW (1971) Journal of Applied Mechanics, 38(2), 448-454. Wave Propagation in Viscoelastic Laminates.

of viscoelastic constitutive relations of the differential form; it is these equations which yield a viscoelastic development which closely mirrors Sun's [8] elastic derivation.

THE ENERGY PRINCIPLE

As Sun [8] does in the development of an elastic laminated beam theory, the first task in deriving a viscoelastic laminated beam theory is to formulate energies for individual viscoelastic layers in terms of the Timoshenko [12] beam theory. In the past, Lee [13] developed viscoelastic Timoshenko beam equations for viscoelastic extensional strain but the shear strain was left elastic. Pan [14] extended the analysis to include viscoelastic shear strains. The current objective is to develop the viscoelastic Timoshenko beam equations in a form more suitable to the development of a viscoelastic composite beam theory. A first goal will be the development of a single layer energy principle suitable for a direct application in the derivation of a multilayer energy principle.

The development of an approximate theory such as for laminated elastic plates has originally been a two step procedure. In the first

---

[8] C. T. SUN (1971) Journal of Applied Mechanics, 38, 947-954 Microstructure Theory for a Composite Beam.

[12] S. P. TIMOSHENKO (1922) Philosophical Magazine, Ser. 6, Vol. 43, 125-131. On the Transverse Vibrations of Bars of Uniform Cross-Section.

[13] H. C. LEE (1960) Journal of Applied Mechanics, 27, 551-556. Forced Lateral Vibration of a Uniform Cantilever Beam with Internal and External Damping.

[14] H. PAN (1966) Jour. Eng. Mech. Div., Proc. Amer. Soc. Civil Eng., 213-234. Vibration of a Viscoelastic Timoshenko Beam.

instance, the Mindlin plate theory [15] in its first order approximation was utilized to develop a continuum theory for laminated composites. Then to obtain a laminated plate theory a first order approximation is made on those variables in the continuum theory which came from the zero order part of the Mindlin plate theory and a zero order approximation is made on the variables which came from the first order part of the Mindlin theory as in Sun [4] and Thomas [5] - this explanation will become clear shortly. Now in developing an elastic laminated beam theory, Sun [8] has made both of these approximations simultaneously to obtain a flexure theory for laminated beams. Actually, Thomas [9] has shown that the flexure beam theory is directly obtainable from the existing flexure plate theory.

The current objective is to immediately derive a viscoelastic laminated beam theory and to not have to develop a viscoelastic laminated continuum theory first. In making the various zero and first

---

[4] C. T. SUN (1971) Journal of Applied Mechanics, 38, 231-238. Theory of Laminated Plates.

[5] C. R. THOMAS (1972) Journal of Sound and Vibration, 25(3), 407-431. Velocity Corrected Theory of Laminated Plates Applied to Free Plate Strip Vibrations.

[8] C. T. SUN (1971) Journal of Applied Mechanics, 38, 947-954. Microstructure Theory for a Composite Beam.

[9] C. R. THOMAS (1973) Watervliet Arsenal Technical Report, R-WV-T-6-45-73. Flexure Equations of Motion for Laminated Composite Beams.

[15] R. D. MINDLIN (1955) Signal Corps Engineering Laboratories, Fort Monmouth, New Jersey (AD-88471). An Introduction to the Mathematical Theory of Vibrations of Elastic Plates.

order expansions of displacement, terms which lead to an extension theory are also maintained since the second expansion of extensional displacements leads to a flexure term. The first order displacements which will result in the Timoshenko beam equations [12] for flexure as well as an extensional equation for beams are

$$v(y,z,t) = \bar{v}(y,t) - z\phi(y,t)$$
$$w(y,z,t) = \bar{w}(y,t) - z\Phi(y,t). \tag{1}$$

the zero order terms in (1) are $\bar{v}$ and $\bar{w}$ and a first order expansion of these two displacements results in the expressions

$$\bar{v}(y,t) = v_\alpha^k(y,t) - z_\alpha^k\, \psi_\alpha(y,t)$$
$$\bar{w}(y,t) = w_\alpha^k(y,t) - z_\alpha^k\, \Psi_\alpha(y,t) \tag{2}$$

where the subscript $\alpha = 1, 2$ will later denote whether a stiff or soft laminated beam layer is indicated and the superscript k which layer pair is indicated. While absolutely necessary at this point, the notation in (2) jumps into the laminate notation while seeming to be at the single layer stage of development. See Sun, Achenbach, and Herrmann [1] or Sun [8] if clarification is required.

Combining equations (1) and (2) and extracting only those terms which result in flexural motion results in the displacement relations

---

[1] C. T. SUN, J. D. ACHENBACH and G. HERRMANN (1968) Journal of Applied Mechanics, 35, 467. Continuum Theory for a Laminated Medium.

[8] C. T. SUN (1971) Journal of Applied Mechanics, 38, 947-954. Microstructure Theory for a Composite Beam.

[12] S. P. TIMOSHENKO (1922) Philosophical Magazine, Ser 6, Vol 43, 125-131. On the Transverse Vibrations of Bars of Uniform Cross-Section.

$$v(y,z,t) = -z_{\alpha}^{k}\psi_{\alpha}(y,t) - z\phi(y,t)$$

$$w(y,z,t) = w_{\alpha}^{k}(y,t) \tag{3}$$

where $\psi_{\alpha}(y,t)$ represents the gross rotation in the laminated beam, $w_{\alpha}^{k}(y,t)$ represents the transverse deflection, and $\phi(y,t)$ represents the individual layer rotation. The various displacements and rotations on the right side of (1) refer to the individual layers which will eventually make up the laminated beam. The various displacements and rotations on the right side of (2) represent the reduction from a laminated continuum theory to a laminated beam theory; thus, from continuity of displacement and rotation at laminate interfaces, it is clear that the notation may be simplified to $w(y,t) = w_{\alpha}^{k}(y,t)$ and $\psi(y,t) = \psi_{\alpha}(y,t)$ for $\alpha = 1, 2$ and for all values of k. Hence with these notational simplifications in mind, the final form of the first order flexure displacement expansion is

$$v(y,z,t) = -z_{\alpha}^{k}\psi(y,t) - z\phi(y,t)$$

$$w(y,z,t) = w(y,t) \tag{4}$$

where these equations are valid only when eventually utilized in developing a laminated beam theory. Equations (4) may be reduced to those for a homogeneous or single layered beam by setting $\psi(y,t) = 0$; this being done, equations (4) reduce to those given by Brunelle [16] for flexure of a beam.

[16] E. J. BRUNELLE (1970) J. Composite Materials, 4, 404-416. The Statics and Dynamics of a Transversely Isotropic Timoshenko Beam.

The non-zero strain-displacement relations are

$$\varepsilon_y = \frac{\partial v}{\partial y}$$

$$\varepsilon_{yz} = \frac{1}{2}\left[\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right] \quad .$$

The non-zero stress equations of motion which pertain to the problem are

$$\sigma_{yz,y} = \rho\ddot{w}$$

$$\sigma_{y,y} + \sigma_{yz,z} = \rho\ddot{v} \quad . \tag{6}$$

From the appendix and equations (A-17) the constitutive equations for a special case of the standard linear model are

$$(1 + C\frac{\partial}{\partial t})\sigma_{yz} = (2kG + 2k^{*}G^{*}\frac{\partial}{\partial t})\ \varepsilon_{yz}$$

$$(1 + C\frac{\partial}{\partial t})\sigma_{y} = (E + E^{*}\frac{\partial}{\partial t})\ \varepsilon_{y} \tag{7}$$

where shear correction coefficients k and k^{*} have now been introduced in a manner similar to that of Timoshenko [12] and Mindlin and Deresiewicz [17].

The procedure involved in deriving the theory will be to manipulate the left sides of equations (6) until they are of the form of the left sides of equations (7). Thus, taking the first time derivatives of (6) and multiplying by the viscoelastic constant C results in the

[12] S. P. TIMOSHENKO (1922) Philosophical Magazine, Ser 6, Vol 43, 125-131. On the Transverse Vibrations of Bars of Uniform Cross-Section.

[17] R. D. MINDLIN and H. DERESIEWICZ (1953) Columbia University Technical Report No. 10. Timoshenko's Shear Coefficient for Flexural Vibrations of Beams.

equations

$$C\,\dot{\sigma}_{yz,y} = \rho C\,\dddot{w}$$

$$C\,\dot{\sigma}_{y,y} + C\,\dot{\sigma}_{yz,z} = \rho C\,\dddot{v} \qquad (8)$$

which when added to their counterparts in equation (6) become

$$\sigma_{yz,y} + C\,\dot{\sigma}_{yz,y} = \rho\ddot{w} + \rho C\,\dddot{w}$$

$$\sigma_{y,y} + C\,\dot{\sigma}_{y,y} + \sigma_{yz,z} + C\,\dot{\sigma}_{yz,z} = \rho\ddot{v} + \rho C\,\dddot{v} \quad . \qquad (9)$$

Multiplying the first equation of (8) by $\dot{w}$ and the second equation by $\dot{v}$, integrating over the beam volume and time, and finally adding the final answers results in the equation

$$\int_A \int_0^\ell \int_0^t \left[\begin{array}{l} (\sigma_{yz,y} + C\,\dot{\sigma}_{yz,y})\dot{w} + (\sigma_{y,y} + C\,\dot{\sigma}_{y,y})\dot{v} \\ + (\sigma_{yz,z} + C\,\dot{\sigma}_{yz,z})\dot{v} \end{array} \right] dA\,dy\,dt$$

$$= \int_A \int_0^\ell \int_0^t \rho \left[\ddot{v}\,\dot{v} + C\,\dddot{v}\,\dot{v} + \ddot{w}\,\dot{w} + C\,\dddot{w}\,\dot{w}) \right] dA\,dy\,dt \qquad (10)$$

After several integrations by parts, equation (10) may be expressed as

$$\int_A \int_0^t \left[(\sigma_{yz} + C\,\dot{\sigma}_{yz})\dot{w} + (\sigma_y + C\,\dot{\sigma}_y)\dot{v} \right]_0^\ell dA\,dt$$

$$+ \int_A \int_0^\ell \int_0^t \frac{d}{dz} \left[(\sigma_{yz} + C\,\dot{\sigma}_{yz})\dot{v} \right] dA\,dy\,dt$$

$$
-\int_A \int_0^{\ell} \int_0^t \left[\begin{array}{l} (\sigma_{yz} + C\,\dot{\sigma}_{yz})\,(\frac{\partial \dot{w}}{\partial y} + \frac{\partial \dot{v}}{\partial z}) \\ +(\sigma_y + C\,\dot{\sigma}_y)\,\frac{\partial \dot{v}}{\partial y} \end{array} \right] dA\,dy\,dt
$$

$$
= \int_A \int_0^{\ell} \int_0^t \rho \left[(\ddot{w} + C\,\dddot{w})\,\dot{w} + (\ddot{v} + C\,\dddot{v})\,\dot{v} \right] dA\,dy\,dt \qquad (11)
$$

it is immediately clear that

$$
\int_A \int_0^{\ell} \int_0^t \frac{d}{dz} \left[(\sigma_{yz} + C\,\dot{\sigma}_{yz})\,\dot{v} \right] dA\,dy\,dt = 0 \qquad (12)
$$

since both beam surfaces are stress free and that

$$
\int_A \int_0^t \left[(\sigma_{yz} + C\,\dot{\sigma}_{yz})\,\dot{w} + (\sigma_y + C\,\dot{\sigma}_y)\,\dot{v} \right]_0^{\ell} dA\,dt = 0 \qquad (13)
$$

since the boundary terms will be satisfied at the beam ends. Applying equations (5) and (7) to equation (11) and taking into account equations (12) and (13) results in

$$
\int_A \int_0^{\ell} \int_0^t \left[\begin{array}{l} (2kG\varepsilon_{yz} + 2k^*G^*\dot{\varepsilon}_{yz})(2\dot{\varepsilon}_{yz}) \\ +(E\varepsilon_y + E^*\dot{\varepsilon}_y)\dot{\varepsilon}_y \end{array} \right] dA\,dy\,dt
$$

$$
= \int_A \int_0^{\ell} \int_0^t \rho[(\ddot{w} + C\,\dddot{w})\dot{w} + (\ddot{v} + C\,\dddot{v})\dot{v}]\ dA\,dy\,dt \quad . \qquad (14)
$$

But, from the chain rule of partial differentiation it is clear that

$$
\frac{\partial}{\partial t}[\varepsilon^2] = \varepsilon\dot{\varepsilon} + \dot{\varepsilon}\varepsilon \qquad (15)
$$

or that

$$
\varepsilon\dot{\varepsilon} = \frac{1}{2}\,\frac{d}{dt}\,(\varepsilon^2) \quad . \qquad (16)
$$

Similarly, the fact that an indefinite integral can be defined as a definite integral with a variable upper limit

$$\int g(t)dt = \int_a^t g(t)dt + \text{const.} \tag{17}$$

immediately results, after taking a time derivative of both sides, in the equation

$$\frac{d}{dt}\int_a^t g(t)dt = g(t) \tag{18}$$

which for $g(t) = \dot{\varepsilon}^2$ results in the relationship

$$\dot{\varepsilon}^2 = \frac{d}{dt}\int_0^t (\dot{\varepsilon})^2 dt \quad . \tag{19}$$

A direct application of relations (16) and (19) to equation (14) with an introduction of equations (4) and (5) results in the equation

$$\int_0^\ell \int_0^t \frac{d}{dt}\left[\begin{array}{l} \frac{1}{2}AkG(\frac{\partial w}{\partial y} - \phi)^2 + Ak^*G^*\int_0^t (\frac{\partial \dot{w}}{\partial y} - \dot{\phi})^2 d\tau \\ \\ + \frac{AE}{2}(z_\alpha^k)^2(\frac{\partial \psi}{\partial y})^2 + AE^*\int_0^t (z_\alpha^k)^2(\frac{\partial \dot{\psi}}{\partial y})^2 d\tau \\ \\ + \frac{EI}{2}(\frac{\partial \phi}{\partial y})^2 + E^*I\int_0^t (\frac{\partial \dot{\phi}}{\partial y})^2 d\tau \end{array}\right] dy\,dt$$

$$+ \int_0^\ell \int_0^t \frac{\rho}{2}\frac{d}{dt}\left[\begin{array}{l} A\dot{w}^2 - 2AC\int_0^t \dot{w}^2 d\tau + A(z_\alpha^k)^2\dot{\psi}^2 \\ \\ + I\dot{\phi}^2 - 2AC\int_0^t (z_\alpha^k)^2\dot{\psi}^2 d\tau \\ \\ - 2IC\int_0^t \dot{\phi}^2 d\tau \end{array}\right] dy\,dt = 0 \tag{20}$$

after an integration over the beam area where

$$A = bd$$

$$I = \frac{bd^3}{12} \tag{21}$$

with b being the beam width and d being the beam thickness.

Following Anderson [18], a conservation law is sought in the existence of a quantity H such that

$$H = \text{constant}, \tag{22}$$

such that obviously

$$\frac{dH}{dt} = 0 \tag{23}$$

where

$$H = T + U + V \tag{24}$$

with the quantities T, U, and V being called the kinetic energy, the potential energy, and the dissipation energy. From a comparison of equations (20), (23), and (24) it is clear that the various energies may be defined as

$$T = \int_0^\ell \int_0^t T^* \, dy \, dt$$

$$U = \int_0^\ell \int_0^t U^* \, dy \, dt$$

$$V = \int_0^\ell \int_0^t V^* \, dy \, dt \tag{25}$$

[18] G. L. ANDERSON (1975) Journal of Sound and Vibration, 39(1), 55-76. Stability of a Rotating Cantilever Subjected to Dissipative, Aerodynamic, and Transverse Follower Forces.

and from equation (20) it is clear that these energies are

$$T^* = \frac{\rho}{2}\,[A\dot{w}^2 + A(z_\alpha^k)^2\dot{\psi}^2 + I\dot{\phi}^2]$$

$$U^* = \frac{1}{2}\,AkG(\frac{\partial w}{\partial y} - \phi)^2 + \frac{AE}{2}(z_\alpha^k)^2(\frac{\partial\psi}{\partial y})^2 + \frac{EI}{2}(\frac{\partial\phi}{\partial y})^2$$

$$V^* = \int_0^\tau \left[\begin{array}{l} Ak^*G^*(\frac{\partial\dot{w}}{\partial y} - \dot{\phi})^2 + E^*I(\frac{\partial\dot{\phi}}{\partial y})^2 \\ + AE^*(z_\alpha^k)^2(\frac{\partial\dot{\psi}}{\partial y})^2 - \rho AC\,\ddot{w}^2 \\ - \rho AC\,(z_\alpha^k)^2\ddot{\psi}^2 - \rho IC\,\ddot{\phi}^2 \end{array} \right] d\tau. \qquad (26)$$

THE LAMINATED BEAM THEORY

The laminated beam, Figure 1, is composed of a number of alternating plane, parallel layers of two homogeneous, isotropic viscoelastic materials which are respectively termed the reinforcing layer and the matrix layer. The reinforcing layer is the stiffer of the two layer combination and is indicated by the subscript "1" while the softer matrix layer is indicated by the subscript "2". The elastic constants, the viscoelastic constants, the layer density, and the thickness for the reinforcing and matrix layers respectively are E_1, G_1, E_1^*, G_1^*, C_1, ρ_1, d_1 and E_2, G_2, E_2^*, G_2^*, C_2, ρ_2, d_2.
The basic variables involved are w, the transverse deflection; ψ, the gross rotation; ϕ_1, the rotation of the stiff layer; and ϕ_2, the rotation of the soft layer. The midplane positions for the kth pair of neighboring reinforcing and matrix layers are y_1^k and y_2^k respectively

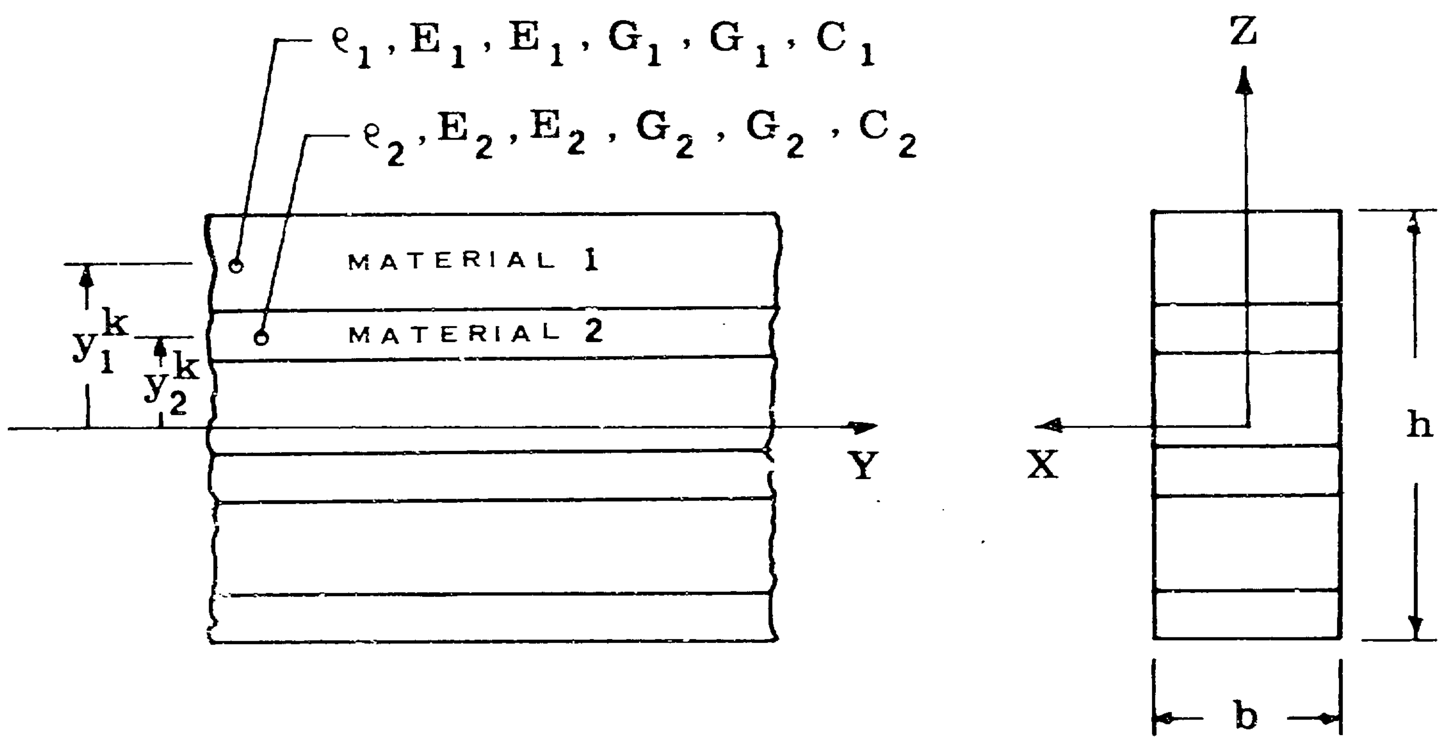

Figure 1 - The Beam Coordinates

as indicated in Figure 1, with the layer midplanes taken perpendicular to the z-axis. The width of the beam is b and the total or gross thickness is h.

From equation (26), the kinetic, potential, and dissipative energies in the individual layers are

$$T_\alpha^{*k} = \frac{\rho_\alpha}{2}[A_\alpha \dot{w}^2 + A_\alpha (z_\alpha^k)^2 \dot{\psi}^2 + I_\alpha \dot{\phi}_\alpha^2]$$

$$U_\alpha^{*k} = \frac{1}{2} A_\alpha k_\alpha G_\alpha (\frac{\partial w}{\partial y} - \phi_\alpha)^2 + \frac{A_\alpha E_\alpha}{2}(z_\alpha^k)^2 (\frac{\partial \psi}{\partial y})^2 + \frac{E_\alpha I_\alpha}{2}(\frac{\partial \phi_\alpha}{\partial y})^2$$

$$V_\alpha^{*k} = \int_0^\tau \begin{bmatrix} A_\alpha k_\alpha^* G_\alpha^* (\frac{\partial \dot{w}}{\partial y} - \dot{\phi}_\alpha)^2 + E_\alpha^* I_\alpha (\frac{\partial \dot{\phi}_\alpha}{\partial y})^2 \\ + A_\alpha E_\alpha^* (z_\alpha^k)^2 (\frac{\partial \dot{\psi}}{\partial y})^2 - \rho_\alpha A_\alpha C_\alpha \ddot{w}^2 \\ - \rho_\alpha A_\alpha C_\alpha (z_\alpha^k)^2 \ddot{\psi}^2 - \rho_\alpha I_\alpha C_\alpha \ddot{\phi}_\alpha^2 \end{bmatrix} d\tau, \tag{27}$$

where α = 1,2 respectively gives the reinforcing and matrix layer energies.

Now, the three energies are summed over the n layer pairs to determine the total energies for the composite beam

$$T^* = \sum_{k=1}^{k=n} (T_1^{*k} + T_2^{*k})$$

$$U^* = \sum_{k=1}^{k=n} (U_1^{*k} + U_2^{*k})$$

$$V^* = \sum_{k=1}^{k=n} (V_1^{*k} + V_2^{*k}) \; . \tag{28}$$

It is now convenient to convert the discrete system (28) to a continuous system by utilization of a smoothing operation, that is to replace the summations in (28) by weighted integrations over the thickness variable z.

The result of the smoothing operation is the energies

$$T^* \approx \int_{-h/2}^{h/2} \frac{1}{(d_1+d_2)} (T_1^* + T_2^*) \, dz$$

$$U^* \approx \int_{-h/2}^{h/2} \frac{1}{(d_1+d_2)} (U_1^* + U_2^*) \, dz$$

$$V^* \approx \int_{-h/2}^{h/2} \frac{1}{(d_1+d_2)} (V_1^* + V_2^*) \, dz \tag{29}$$

where after smoothing

$$z = z_1^k = z_2^k \; . \tag{30}$$

Carrying out the integrations in (29) in terms of (27) and taking into account (30) results in the energies

$$T^* = \frac{1}{2}(\rho_1 A_1 + \rho_2 A_2)\frac{h}{(d_1+d_2)}\dot{w}^2 + \frac{1}{24}(\rho_1 A_1 + \rho_2 A_2)\frac{h^3}{(d_1+d_2)}\dot{\psi}^2$$

$$+ \frac{1}{2}\rho_1 I_1 \frac{h}{(d_1+d_2)}\dot{\phi}_1^2 + \frac{1}{2}\rho_2 I_2 \frac{h}{(d_1+d_2)}\dot{\phi}_2^2$$

$$U^* = \frac{1}{2} A_1 k_1 G_1 \frac{h}{(d_1+d_2)}\left(\frac{\partial w}{\partial y} - \phi_1\right)^2 + \frac{1}{2} A_2 k_2 G_2 \frac{h}{(d_1+d_2)}\left(\frac{\partial w}{\partial y} - \phi_2\right)^2$$

$$+ \frac{1}{24}(A_1 E_1 + A_2 E_2)\frac{h^3}{(d_1+d_2)}\left(\frac{\partial \psi}{\partial y}\right)^2 + \frac{1}{2} E_1 I_1 \frac{h}{(d_1+d_2)}\left(\frac{\partial \phi_1}{\partial y}\right)^2$$

$$+ \frac{1}{2} E_2 I_2 \left(\frac{\partial \phi_2}{\partial y}\right)^2$$

$$V^* = \int_0^t \left[\begin{aligned} & A_1 k_1^* G_1^* \frac{h}{(d_1+d_2)}\left(\frac{\partial \dot{w}}{\partial y} - \dot{\phi}_1\right)^2 + A_2 k_2^* G_2^* \frac{h}{(d_1+d_2)}\left(\frac{\partial \dot{w}}{\partial y} - \dot{\phi}_2\right)^2 \\ & + \frac{1}{12}(A_1 E_1^* + A_2 E_2^*)\frac{h^3}{(d_1+d_2)}\left(\frac{\partial \dot{\psi}}{\partial y}\right)^2 + E_1^* I_1 \frac{h}{(d_1+d_2)}\left(\frac{\partial \dot{\phi}_1}{\partial y}\right)^2 \\ & + E_2^* I_2 \frac{h}{(d_1+d_2)}\left(\frac{\partial \dot{\phi}_2}{\partial y}\right)^2 - (\rho_1 A_1 C_1 + \rho_2 A_2 C_2)\frac{h}{(d_1+d_2)}\ddot{w}^2 \\ & - \frac{1}{12}(\rho_1 A_1 C_1 + \rho_2 A_2 C_2)\frac{h^3}{(d_1+d_2)}\ddot{\psi}^2 - \rho_1 I_1 C_1 \frac{h}{(d_1+d_2)}\ddot{\phi}_1^2 \\ & - \rho_2 A_2 C_2 \frac{h}{(d_1+d_2)}\ddot{\phi}_2^2 \end{aligned}\right] d\tau. \tag{31}$$

At this point, continuity of displacement at the interface of the kth pair of layers must be considered. Applying equation [illegible] to a multilayer beam results in the equation

$$v_\alpha(y,z,t) = -z_\alpha^k(y,t) - z\phi_\alpha(y,t) \tag{32}$$

and with the aid of Figure 2 it is clear that

$$v_1 = -z_1^k\psi + \frac{d_1}{2}\phi_1$$

$$v_2 = -z_2^k\psi - \frac{d_2}{2}\phi_2 \tag{33}$$

at the interface between layers 1 and 2. It is also clear from Figure 2 that

$$z_2^k = z_1^k - \frac{1}{2}(d_1 + d_2) \tag{34}$$

and that equations (33) describe the same interface such that

$$v_1 = v_2 \quad . \tag{35}$$

From equations (35) applied to equations (33) it is clear that the continuity condition is

$$\psi = \eta\phi_1 + (1-\eta)\phi_2 \tag{36}$$

where

$$\eta = \frac{d_1}{(d_1+d_2)} \quad , \qquad (1-\eta) = \frac{d_2}{(d_1+d_2)} \quad . \tag{37}$$

Following Sun [8], the variable ϕ_2 is eliminated such that

$$\phi_2 = \frac{\psi-\eta\phi}{(1-\eta)} \tag{38}$$

where for convenience the notation $\phi = \phi_1$ has been introduced.

Expression (38) is directly substituted into equations (31) and the dimensionless variable

$$\xi = h/(d_1 + d_2) \tag{39}$$

[8] C. T. SUN (1971) Journal of Applied Mechanics, 38, 947-954. Microstructure Theory for a Composite Beam.

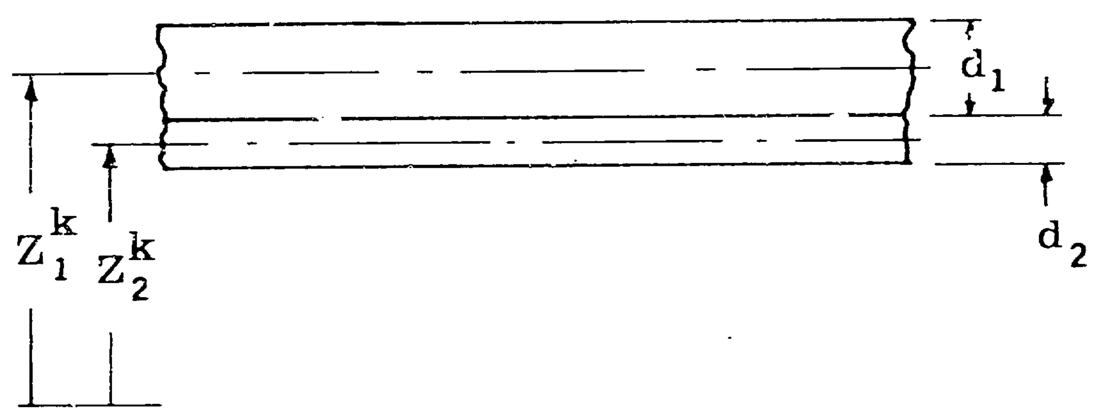

Figure 2 - The Layer Midplanes

is introduced to yield the energy expressions

$$T^* = \frac{1}{2}\xi(\rho_1 A_1 + \rho_2 A_2)\,\dot{w}^2 + \frac{1}{2}I_b[\eta\rho_1 + (1-\eta)\rho_2]\dot{\psi}^2$$

$$+ \frac{1}{2}\xi\rho_1 I_1\dot{\phi}^2 + \frac{1}{2}\xi\rho_2 I_2\left(\frac{\dot{\psi}}{(1-\eta)} - \frac{\eta}{(1-\eta)}\dot{\phi}\right)^2$$

$$U^* = \frac{1}{2}\xi A_1 k_1 G_1\left(\frac{\partial w}{\partial y} - \phi\right)^2 + \frac{1}{2}\xi A_2 k_2 G_2\left(\frac{\partial w}{\partial y} - \frac{\psi}{(1-\eta)} + \frac{\eta}{(1-\eta)}\phi\right)^2$$

$$+ \frac{I_b}{2}(\eta E_1 + (1-\eta)E_2)\left(\frac{\partial\psi}{\partial y}\right)^2 + \frac{1}{2}E_1 I_1\left(\frac{\partial\phi}{\partial y}\right)^2$$

$$+ \frac{1}{2}\xi E_2 I_2\left(\frac{1}{(1-\eta)}\frac{\partial\psi}{\partial y} - \frac{\eta}{(1-\eta)}\frac{\partial\phi}{\partial y}\right)^2$$

$$V^* = \int_0^t \left[\begin{aligned} &\xi A_1 k_1^* G_1^*\left(\frac{\partial\dot{w}}{\partial y} - \dot{\phi}\right)^2 + \xi A_2 k_2^* G_2^*\left(\frac{\partial\dot{w}}{\partial y} - \frac{\dot{\psi}}{(1-\eta)} + \frac{\eta\dot{\phi}}{(1-\eta)}\right)^2 \\ &+ I_b[\eta E_1^* + (1-\eta)E_2^*]\left(\frac{\partial\dot{\psi}}{\partial y}\right)^2 + \xi E_1^* I_1\left(\frac{\partial\dot{\phi}}{\partial y}\right)^2 \\ &+ \xi E_2^* I_2\left(\frac{1}{(1-\eta)}\frac{\partial\dot{\psi}}{\partial y} - \frac{\eta}{(1-\eta)}\frac{\partial\dot{\phi}}{\partial y}\right)^2 - \xi(\rho_1 A_1 C_1 + \rho_2 A_2 C_2)\ddot{w} \\ &- I_b[\eta\rho_1 C_1 + (1-\eta)\rho_2 C_2]\ddot{\psi}^2 - \xi\rho_1 I_1 C_1\,\ddot{\phi}^2 \\ &- \xi\rho_2 I_2 C_2\left(\frac{1}{(1-\eta)}\ddot{\psi} - \frac{\eta}{(1-\eta)}\ddot{\phi}\right)^2 \end{aligned}\right] d\tau \qquad (40)$$

where

$$I_b = \frac{bh^3}{12} \quad . \qquad (41)$$

Now, all the squares of the various sums in equations (40) are expanded out to yield the final forms of the energy expressions as

$$T^* = \frac{1}{2}\xi a_4\dot{w}^2 + \frac{1}{2}\xi a_9\dot{\psi}^2 + \frac{1}{2}\xi a_{13}\dot{\phi}^2 - \xi a_{10}\dot{\psi}\dot{\phi}$$

$$U^* = \frac{1}{2}\xi a_1\left(\frac{\partial w}{\partial y}\right)^2 - \xi a_3\phi\frac{\partial w}{\partial y} + \frac{1}{2}\xi a_{12}\phi^2 - \xi a_2\psi\frac{\partial w}{\partial y}$$

$$+ \frac{1}{2}\xi a_6\psi^2 - \xi a_8\phi\psi + \frac{1}{2}\xi a_5\left(\frac{\partial \psi}{\partial y}\right)^2 + \frac{1}{2}\xi a_{11}\left(\frac{\partial \phi}{\partial y}\right)^2$$

$$- \xi a_7\frac{\partial \psi}{\partial y}\,\frac{\partial \phi}{\partial y}$$

$$V^* = \int_0^t \left[\begin{array}{l} \xi b_1\left(\frac{\partial \dot{w}}{\partial y}\right)^2 - 2\xi b_3\dot{\phi}\frac{\partial \dot{w}}{\partial y} + \xi b_{12}\dot{\phi}^2 - 2\xi b_2\dot{\psi}\frac{\partial \dot{w}}{\partial y} + \xi b_6\dot{\psi}^2 \\ -2\xi b_8\dot{\phi}\dot{\psi} + \xi b_5\left(\frac{\partial \dot{\psi}}{\partial y}\right)^2 + \xi b_{11}\left(\frac{\partial \dot{\phi}}{\partial y}\right)^2 - 2\xi b_7\frac{\partial \dot{\psi}}{\partial y}\,\frac{\partial \dot{\phi}}{\partial y} \\ -\xi b_4\ddot{w}^2 - \xi b_9\ddot{\psi}^2 - \xi b_{13}\ddot{\phi}^2 + 2\xi b_{10}\ddot{\psi}\ddot{\phi} \end{array}\right] d\tau \quad (42)$$

where the constants a_i are

$$a_1 = A_1k_1G_1 + A_2k_2G_2$$

$$a_2 = A_2k_2G_2/(1-\eta)$$

$$a_3 = A_1k_1G_1 - \eta A_2k_2G_2/(1-\eta)$$

$$a_4 = \rho_1A_1 + \rho_2A_2$$

$$a_5 = \frac{I_b}{\xi}[\eta E_1 + (1-\eta)E_2] + \frac{E_2I_2}{(1-\eta)^2}$$

$$a_6 = A_2k_2G_2/(1-\eta)^2 = a_2/(1-\eta)$$

$$a_7 = \frac{\eta}{(1-\eta)^2} E_2 I_2$$

$$a_8 = \frac{\eta}{(1-\eta)^2} A_2 k_2 G_2 = \eta a_6$$

$$a_9 = \frac{I_b}{\xi}[\eta\rho_1 + (1-\eta)\rho_2] + \frac{\rho_2 I_2}{(1-\eta)^2}$$

$$a_{10} = \frac{\eta}{(1-\eta)^2} \rho_2 I_2$$

$$a_{11} = E_1 I_1 + \frac{\eta^2}{(1-\eta)^2} E_2 I_2$$

$$a_{12} = A_1 k_1 G_1 + \frac{\eta^2}{(1-\eta)^2} A_2 k_2 G_2$$

$$a_{13} = \rho_1 I_1 + \frac{\eta^2}{(1-\eta)^2} \rho_2 I_2 \qquad (43)$$

which correspond to the elastic constants given by Sun [8] for elastic laminated beams and where the constants b_i are

$$b_1 = A_1 k_1^* G_1^* + A_2 k_2^* G_2^*$$

$$b_2 = A_2 k_2^* G_2^* / (1-\eta)$$

$$b_3 = A_1 k_1^* G_1^* - \frac{\eta}{(1-\eta)} A_2 k_2^* G_2^*$$

$$b_4 = \rho_1 A_1 C_1 + \rho_2 A_2 C_2$$

$$b_5 = \frac{I_b}{\xi}[\eta E_1^* + (1-\eta) E_2^*] + \frac{E_2^* I_2}{(1-\eta)^2}$$

$$b_6 = A_2 k_2^* G_2^* / (1-\eta)^2 = b_2/(1-\eta)$$

$$b_7 = \frac{\eta}{(1-\eta)^2} E_2^* I_2$$

---

[8] C. T. SUN (1971) Journal of Applied Mechanics, 38, 947-954
Microstructure Theory for a Composite Beam.

$$b_8 = \frac{\eta}{(1-\eta)^2} A_2 k_2^* G_2^* = \eta b_6$$

$$b_9 = \frac{I_b}{\xi}[\eta\rho_1 C_1 + (1-\eta)\rho_2 C_2] + \frac{1}{(1-\eta)^2} \rho_2 I_2 C_2$$

$$b_{10} = \frac{\eta}{(1-\eta)^2} \rho_2 I_2 C_2$$

$$b_{11} = E_1^* I_1 + \frac{\eta^2}{(1-\eta)^2} E_2^* I_2$$

$$b_{12} = A_1 k_1^* G_1^* + \frac{\eta^2}{(1-\eta)^2} A_2 k_2^* G_2^*$$

$$b_{13} = \rho_1 I_1 C_1 + \frac{\eta^2}{(1-\eta)^2} \rho_2 I_2 C_2 \qquad (44)$$

which correspond to the viscoelastic contribution of the current analysis of viscoelastic laminated beams. It should be noted that the author [19] has evaluated viscoelastic shear correction constants in another paper and based on this evaluation it is clear that $k_1 = k_2 = k_1^* = k_2^* = \pi^2/12$.

Now, from equations (22-25) in conjunction with equations (42) it is easy to form energy principle (23), that is dH/dt=0, which upon various integrations by parts and a gathering of common factors of $\dot{w}, \dot{\psi}$, and ϕ results in equation (23) becoming

[19] C. R. THOMAS (1976) Watervliet Arsenal Technical Report, WVT-TR-76009 . Simple Thickness Modes and Shear Correction Coefficients for Viscoelastic Timoshenko Beams.

$$\frac{dH}{dt} = \int_0^t \int_0^{\ell} \dot{w} \left[-a_1\frac{\partial^2 w}{\partial y^2} + a_2\frac{\partial\psi}{\partial y} + a_3\frac{\partial\phi}{\partial y} + a_4\ddot{w} - b_1\frac{\partial^2\dot{w}}{\partial y^2} + b_2\frac{\partial\dot{\psi}}{\partial y} + b_3\frac{\partial\dot{\phi}}{\partial y} + b_4\dddot{w} \right] dtdy$$

$$+ \int_0^t \int_0^{\ell} \dot{\psi} \left[-a_2\frac{\partial w}{\partial y} - a_5\frac{\partial^2\psi}{\partial y^2} + a_6\psi + a_7\frac{\partial^2\phi}{\partial y^2} - a_8\phi + a_9\ddot{\psi} - a_{10}\ddot{\phi} - b_2\frac{\partial\dot{w}}{\partial y} - b_5\frac{\partial^2\dot{\psi}}{\partial y^2} + b_6\dot{\psi} + b_7\frac{\partial^2\dot{\phi}}{\partial y^2} - b_8\dot{\phi} + b_9\dddot{\psi} - b_{10}\dddot{\phi} \right] dtdy$$

$$+ \int_0^t \int_0^{\ell} \dot{\phi} \left[-a_3\frac{\partial w}{\partial y} + a_7\frac{\partial^2\psi}{\partial y^2} - a_8\psi - a_{10}\ddot{\psi} - a_{11}\frac{\partial^2\phi}{\partial y^2} + a_{12}\phi + a_{13}\ddot{\phi} - b_3\frac{\partial\dot{w}}{\partial y} + b_7\frac{\partial^2\dot{\psi}}{\partial y^2} - b_8\dot{\psi} - b_{10}\dddot{\psi} - b_{11}\frac{\partial^2\dot{\phi}}{\partial y^2} + b_{12}\dot{\phi} + b_{13}\dddot{\phi} \right] dtdy$$

$$+ \int_0^t \dot{w} \left[a_1\frac{\partial w}{\partial y} - a_2\psi - a_3\phi + b_1\frac{\partial\dot{w}}{\partial y} - b_2\dot{\psi} - b_3\dot{\phi} \right]_0^{\ell} dt$$

$$+ \int_0^t \dot{\psi} \left[a_5\frac{\partial\psi}{\partial y} - a_7\frac{\partial\phi}{\partial y} + b_5\frac{\partial\dot{\psi}}{\partial y} - b_7\frac{\partial\dot{\phi}}{\partial y} \right]_0^{\ell} dt$$

$$+ \int_0^{t_1} \dot{\phi} \left[- a_7 \frac{\partial\psi}{\partial y} + a_{11} \frac{\partial\phi}{\partial y} - b_7 \frac{\partial\dot{\psi}}{\partial y} + b_{11} \frac{\partial\dot{\psi}}{\partial y} \right]_0^{\ell} dt = 0 \; . \qquad (45)$$

The viscoelastic equations of motion and boundary conditions for laminated beams are now obtained by applying the first lemma of the calculus of variations to equation (45). Thus, the three equations of motion are

$$a_1 \frac{\partial^2 w}{\partial y^2} - a_2 \frac{\partial\psi}{\partial y} - a_3 \frac{\partial\phi}{\partial y} + b_1 \frac{\partial^2 \dot{w}}{\partial y^2} - b_2 \frac{\partial\dot{\psi}}{\partial y} - b_3 \frac{\partial\dot{\phi}}{\partial y} = a_4 \ddot{w} + b_4 \dddot{w}$$

$$a_2 \frac{\partial w}{\partial y} + a_5 \frac{\partial^2\psi}{\partial y^2} - a_6 \psi - a_7 \frac{\partial^2\phi}{\partial y^2} + a_8 \phi + b_2 \frac{\partial\dot{w}}{\partial y} + b_5 \frac{\partial^2\dot{\psi}}{\partial y^2} - b_6 \dot{\psi}$$

$$- b_7 \frac{\partial^2\dot{\phi}}{\partial y^2} + b_8 \dot{\phi} = a_9 \ddot{\psi} - a_{10} \ddot{\phi} + b_9 \dddot{\psi} - b_{10} \dddot{\phi}$$

$$a_3 \frac{\partial w}{\partial y} - a_7 \frac{\partial^2\psi}{\partial y^2} + a_8 \psi + a_{11} \frac{\partial^2\phi}{\partial y^2} - a_{12} \phi + b_3 \frac{\partial\dot{w}}{\partial y} - b_7 \frac{\partial^2\dot{\psi}}{\partial y^2} + b_8 \dot{\psi}$$

$$+ b_{11} \frac{\partial^2\dot{\phi}}{\partial y^2} - b_{12} \dot{\phi} = - a_{10} \ddot{\psi} + a_{13} \ddot{\phi} - b_{10} \dddot{\psi} + b_{13} \dddot{\phi} \qquad (46)$$

and the corresponding boundary conditions are

$$a_1 \frac{\partial w}{\partial y} - a_2 \psi - a_3 \phi + b_1 \frac{\partial\dot{w}}{\partial y} - b_2 \dot{\psi} - b_3 \dot{\phi} = 0,$$

or

$$w = 0 \qquad \text{on} \qquad y = 0, \ell \qquad (47\text{-a})$$

$$a_5 \frac{\partial\psi}{\partial y} - a_7 \frac{\partial\phi}{\partial y} + b_5 \frac{\partial\dot{\psi}}{\partial y} - b_7 \frac{\partial\dot{\phi}}{\partial y} = 0,$$

or

$$\psi = 0 \qquad \text{on} \qquad y = 0, \ell \qquad (47\text{-b})$$

$$a_7\frac{\partial\psi}{\partial y} - a_{11}\frac{\partial\phi}{\partial y} + b_7\frac{\partial\dot{\psi}}{\partial y} - b_{11}\frac{\partial\dot{\phi}}{\partial y} = 0,$$

or

$$\phi = 0 \quad \text{on} \quad y = o,\ell \quad . \tag{47-c}$$

SUMMARY

An energy principle has been formulated for viscoelastic Timoshenko beams according to the standard linear model with the stipulation, and hence additional terms, that the energy principle be utilized in building a viscoelastic laminated beam theory. The Timoshenko model considered has accounted for both viscoelastic extensional and viscoelastic shear strains. To later incorporate the single layer energy principle into the development of a laminated beam theory, a term which accounts for the beam's gross rotation was included in the single layer development.

Using the single layer energies developed, a viscoelastic laminated beam theory composed of a number of alternating, plane, parallel layers of two homogenious, isotropic viscoelastic materials, termed the reinforcing layer and the matrix layer, was derived. In deriving the theory, the individual layer kinetic, potential, and dissipative energies were summed over n layer pairs to obtain the total energy of the composite beam; these results are converted to a continuous system by utilization of a smoothing operation or weighted integration. The number of independent variables in the total composite beam energies is reduced from four to three thru the introduction of a condition for continuity at layer interfaces. A direct application of the energy principle developed to the composite beam energies results in a set of three equations of motion and their corresponding boundary conditions for viscoelastic, laminated composite beams. A future report will discuss a number of meaningful results of both wave propagation and vibration analysis utilizing the equations of motion developed in this report.

REFERENCES.

1. C. T. SUN, J. D. ACHENBACH and G. HERRMANN (1968) Journal of Applied Mechanics, 35, 467. Continuum Theory for a Laminated Medium.
2. J. D. ACHENBACH C. T. SUN and G. HERRMANN (1968) Journal of Applied Mechanics, 35, 689. On the Vibrations of a Laminated Body.
3. C. R. THOMAS (1972) Journal of Sound and Vibration, 23(3), 341-361. Simple Thickness Modes for Laminated Composite Materials.
4. C. T. SUN (1971) Journal of Applied Mechanics, 38, 231-238. Theory of Laminated Plates.
5. C. R. THOMAS (1972) Journal of Sound and Vibration, 25(3), 407-431. Velocity Corrected Theory of Laminated Plates Applied to Free Plate Strip Vibrations.
6. C. R. THOMAS (1973) Journal of Sound and Vibration, 31(2), 195-211. Extensional Vibrations of Simply Supported Composite Plate Strips.
7. C. R. THOMAS (1975) Journal of the Acoustical Society of America, 57(3), 655-659. Flexural and Extensional Vibrations of Simply Supported Laminated Rectangular Plates.
8. C. T. SUN (1971) Journal of Applied Mechanics, 38, 947-954. Microstructure Theory for a Composite Beam.

9. C. R. THOMAS (1973) Watervliet Arsenal Technical Report, R-WV-T-6-45-73. Flexure Equations of Motion for Laminated Composite Beams.

10. R. A. GROT and J. D. ACHENBACH (1970) Acta Mechanica, 9, 245-263. Linear Isothermal Theory for a Viscoelastic Laminated Composite.

11. M. STERN, A. BEDFORD, and C. H. YEW (1971) Journal of Applied Mechanics, 38(2), 448-454. Wave Propagation in Viscoelastic Laminates.

12. S. P. TIMOSHENKO (1922) Philosophical Magazine, Ser 6, Vol 43, 125-131. On the Transverse Vibrations of Bars of Uniform Cross-Section.

13. H. C. LEE (1960) Journal of Applied Mechanics, 27, 551-556. Forced Lateral Vibration of a Uniform Cantilever Beam with Internal and External Damping.

14. H. PAN (1966) Jour. Eng. Mech. Div., Proc. Amer. Soc. Civil Eng., 213-234. Vibration of a Viscoelastic Timoshenko Beam.

15. R. D. MINDLIN (1955) Signal Corps Engineering Laboratories, Fort Monmouth, New Jersey (AD-88471). An Introduction to the Mathematical Theory of Vibrations of Elastic Plates.

16. E. J. BRUNELLE (1970) J. Composite Materials, 4, 404-416. The Statics and Dynamics of a Transversely Isotropic Timoshenko Beam.

17. R. D. MINDLIN and H. DERESIEWICZ (1953) Columbia University Technical Report No. 10. Timoshenko's Shear Coefficient for Flexural Vibrations of Beams.

18. G. L. ANDERSON (1975) Journal of Sound and Vibration, 39(1), 55-76. Stability of a Rotating Cantilever Subjected to Dissipative, Aerodynamic, and Transverse Follower Forces.

19. C. R. THOMAS (1976) Watervliet Arsenal Technical Report, WVT-TR-76009 . Simple Thickness Modes and Shear Correction Coefficients for Viscoelastic Timoshenko Beams.

20. Y. C. FUNG (1965) Prentice-Hall, Inc. "Foundations of Solid Mechanics".

APPENDIX

The present objective is to derive a set of constitutive relations which can be utilized in conjunction with the basic equations for a Timoshenko beam. While constitutive equations may be formulated in either integral or differential form, preliminary work in the direction of formulation of a viscoelastic beam theory for laminated composite materials indicates that the differential form of constitutive relations will be most useful. The differential constitutive relations will be utilized in the present development.

The general form of the differential constitutive equations is adapted from Fung [20] where the stress-strain relations are of the form

$$P_1(D)\sigma'_{ij} = Q_1(D)e'_{ij}$$

$$P_2(D)\sigma_{kk} = Q_2(D)e_{kk} \qquad \text{(A-1)}$$

where $P_i(D)$ and $Q_i(D)$ are given by

$$P_1(D) = \sum_{k=o}^{k=n_1} a_k D^k$$

$$P_2(D) = \sum_{k=o}^{k=n_2} c_k D^k$$

[20] Y. C. FUNG (1965) Prentice-Hall, Inc. "Foundations of Solid Mechanics".

$$Q_1(D) = \sum_{k=o}^{k=m_1} b_k D^k$$

$$Q_2(D) = \sum_{k=o}^{k=m_1} d_k D^k \qquad \text{(A-2)}$$

with D being the time-derivative operator of the form

$$D^i f = \frac{\partial^i f(t)}{\partial t^i} \qquad \text{(A-3)}$$

and where σ'_{ij} and e'_{ij} are the components of the stress and strain deviators

$$\sigma'_{ij} = \sigma_{ij} - \frac{1}{3} \delta_{ij} \sigma_{kk}$$

$$e'_{ij} = e_{ij} - \frac{1}{3} \delta_{ij} e_{kk} \qquad \text{(A-4)}$$

in which σ_{ij} and e_{ij} are the components of stress and strain.

Now, assume equations (A-1) to have the form of the standard linear model

$$(1 + \overline{A} \frac{\partial}{\partial t})\sigma = (\overline{B} + \overline{C} \frac{\partial}{\partial t})\varepsilon \qquad \text{(A-5)}$$

where σ is stress and ε is strain. Comparing the form of (A-5) with equations (A-1) it is clear that to have the form of the standard linear model it must be true that

$$n_1 = m_1 = n_2 = m_2 = 1 \qquad \text{(A-6)}$$

and operators (A-2) in light of (A-6) reduce to

$$P_1(D) = a_0 + a_1 D$$

$$Q_1(D) = b_0 + b_1 D$$

$$P_2(D) = C_0 + C_1 D$$

$$Q_2(D) = d_0 + d_1 D \quad . \qquad \text{(A-7)}$$

As will be subsequently seen, the only non-zero stresses and strains for a Timoshenko beam with its y-axis along the length and its z-axis through the thickness are σ_y and σ_{yz} & ε_y and ε_{yz}. Thus, from equation (A-4) the non-zero stress and strain deviators are

$$\sigma'_y = \frac{2}{3}\sigma_y \quad , \quad \sigma'_{yz} = \sigma_{yz}$$

$$\varepsilon'_y = \frac{2}{3}\varepsilon_y \quad , \quad \varepsilon'_{yz} = \varepsilon_{yz} \quad . \qquad \text{(A-8)}$$

Now, a direct substitution of equations (A-2), (A-7), and (A-8) into equation (A-1) results in

$$[1 + (a_1/a_0)D]\sigma_{yz} = [(b_0/a_0) + (b_1/a_0)D]\varepsilon_{yz}$$

$$[1 + (a_1/a_0)D]\sigma_y = [(b_0/a_0) + (b_1/a_0)D]\varepsilon_y$$

$$[1 + (C_1/C_0)D]\sigma_y = [(d_0/C_0) + (d_1/C_0)D]\varepsilon_y \quad . \qquad \text{(A-9)}$$

There are thus two equations for stress-strain in the y- coordinate

$$D_1\sigma_y = D_2\varepsilon_y$$

$$D_3\sigma_y = D_4\varepsilon_y \quad , \qquad \text{(A-10)}$$

where

$$D_1 = 1-(a_1/a_0)D$$

$$D_2 = (b_0/a_0) + (b_1/a_0)D$$

$$D_3 = 1 + (C_1/C_0)D$$

$$D_4 = (d_0/C_0) + (d_1/C_0)D \quad , \tag{A-11}$$

and they must be combined to form a single constitutive equation

$$2D_1D_3\sigma = (D_2D_3 + D_1D_4)\varepsilon_y \quad . \tag{A-12}$$

Now, from both the right and left sides of equation (A-12) it is clear that the constitutive equation is of the form

$$(1 + \bar{a}D + \bar{b}D^2)\sigma_y = (1 + \bar{C}\ D + \bar{d}D^2)\varepsilon_y \quad , \tag{A-13}$$

but it would now be desireable to have the form of the standard linear model as in equation (A-5), if possible. This can be achieved if the restriction is now made that

$$D_1 = D_3 = 1 + (a_1/a_0)D \tag{A-14}$$

such that equation (A-12) now becomes

$$[1 + (a_1/a_0)D]\sigma_y = \frac{1}{2}[(b_0/a_0 + d_0/C_0) + (b_1/a_0 + d_1/C_0)D]\varepsilon_y \quad . \tag{A-15}$$

As a final step, define the constants

$$C = a_1/a_0$$

$$E = \frac{1}{2}(b_0/a_0 + d_0/C_0)$$

$$E^* = \frac{1}{2}(b_1/a_0 + d_1/C_0)$$

$$2G = b_0/a_0$$

$$2G^* = b_1/a_0 \tag{A-16}$$

with the final form of the constitutive equation thus being

$$(1 + C\frac{\partial}{\partial t})\sigma_{yz} = (2G + 2G^{*}\frac{\partial}{\partial t})\varepsilon_{yz}$$

$$(1 + C\frac{\partial}{\partial t})\sigma_{y} = (E + E^{*}\frac{\partial}{\partial t})\varepsilon_{y} \quad . \tag{A-17}$$